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Crocheting the Hyperbolic Plane

For God's sake, please give it up. Fear it no less than the sensual passion, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life.
—Wolfgang Bolyai urging his son János Bolyai to give up work on hyperbolic geometry

In June of 1997, Daina was in a workshop watching the leader of the workshop, David, helping the participants study ideas of hyperbolic geometry using a paper-and-tape surface in much the same way that one can study ideas of spherical geometry by using the surface of a physical ball. David's hyperbolic plane was then so tattered and fragile that he was afraid to handle it much. Daina immediately began to think: "There must be some way to make a durable model."

David made his first paper hyperbolic plane in the summer of 1978, while on canoe trip on the lakes of Maine, using the scissors on his Swiss Army knife. He had just learned how to do the construction from William Thurston at a workshop at Bates College. This crude paper surface was used in David's geometry classes and workshops (becoming more and more tattered) until 1986, when some high school teachers in a summer program that David was leading collaborated on a new, larger paper-and-tape hyperbolic surface. This second paper-and-tape hyperbolic surface (used in classes and workshops for the next 11 years) was the one that Daina witnessed in use.

Daina experimented with knitting (but the result was not rigid enough) and then settled on crocheting. She perfected her technique during the workshop and crocheted her first small hyperbolic plane; then, while camping in the forests of Pennsylvania, she crocheted more, and we started exploring its uses. In this paper we share how to crochet a hyperbolic plane (and make related paper versions). We also share how we have used it to increase our own understanding of hyperbolic geometry. (What are horocycles? Where does the area formula πr^2 fit in hyperbolic geometry?) We will also prove that the intrinsic geometry of these surfaces is, in fact, (an approximation of) hyperbolic geometry.

But, Wait! you say. Do not many books state that it is impossible to embed the hyperbolic plane isometrically (an *isometry* is a function that preserves all distances) as a complete subset of the Euclidean 3-space? Yes, they do: For popularly written examples, see Robert Osserman's *Poetry of the Universe* [9], page 158, and David Hilbert and S. Cohn-Vossen's *Geometry and the Imagination* [6], page 243. For a detailed discussion and proof, see Spivak's *A Comprehensive Introduction to Differential Geometry* [10], Vol. III, pages 373 and 381.

All of the references are implicitly assuming surfaces embedded with some conditions of differentiability, and refer (implicitly or explicitly) to a 1901 theorem by David Hilbert. Hilbert proved [5] that there is no real analytic isometric embedding of the hyperbolic plane onto a complete subset of 3-space, and his arguments also work to show that there is no isometric embedding whose derivatives up to order four are continuous. Moreover, in 1964, N. V. Efimov ([2] Russian; discussed in English in Tilla Milnor's [8]) extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. However, in 1955, N. Kuiper proved [7] that there *is* an isometric embedding with continuous first derivatives of the hyperbolic plane onto a closed subset of 3-space. For a more detailed discussion of these ideas, see Thurston [11], pages 51–52. The finite surfaces described here can apparently be extended indefinitely, but they appear always not to be differentiably embedded (see Figure 12).

Constructions of Hyperbolic Planes

We describe several different isometric constructions of the hyperbolic plane (or approximations of the hyperbolic plane) as surfaces in 3-space.

The hyperbolic plane from paper annuli

This is the paper-and-tape surface that David learned from William Thurston. It may be constructed as follows: Cut out many identical annular strips of radius ρ, as in Figure 1. (An *annulus* is the region between two concentric circles, and we call an *annular strip* a portion of an annulus cut off by an angle from the center of the circles.) Attach the strips together by attaching the inner circle of one to the outer circle of the other or the straight ends together. (When the straight ends of annular strips are attached together you get annular strips with increasing angles, and eventually the angle will be more than 2π.) The resulting surface is of course only an approximation of the desired surface. The actual annular hyperbolic plane is obtained by letting $\delta \to 0$ while holding ρ fixed. (We show below, in several ways, that this limit exists.) Note that because the surface is constructed the same everywhere (as $\delta \to 0$), it is homogeneous (that is, intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). We will call the results of this construction the *annular hyperbolic plane*. We urge the reader to try this by cutting out a few identical annular strips and taping them together as in Figure 1.

How to crochet the annular hyperbolic plane

If you tried to make your annular hyperbolic plane from paper annuli you certainly realized that it takes a lot of time. Also, later you will have to play with it carefully because it is fragile and tears and creases easily—you may want just to leave it sitting on your desk. But there is a way to get a sturdy model of the hyperbolic plane which you can work and play with as much as you wish. This is the crocheted hyperbolic plane.

To make the crocheted hyperbolic plane, you need just a few very basic crocheting skills. All you need to know is how to make a chain (to start) and how to single crochet. See Figure 2 for a picture of these stitches, which will be described further in the next paragraph.

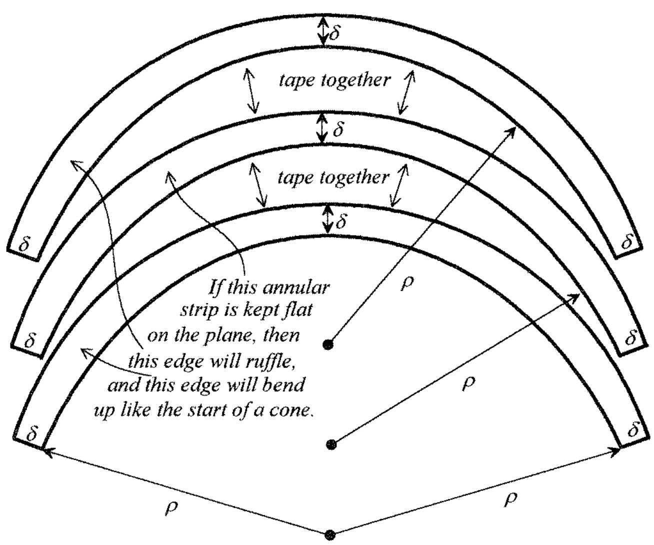

Figure 1. Annular strips for making an annular hyperbolic plane.

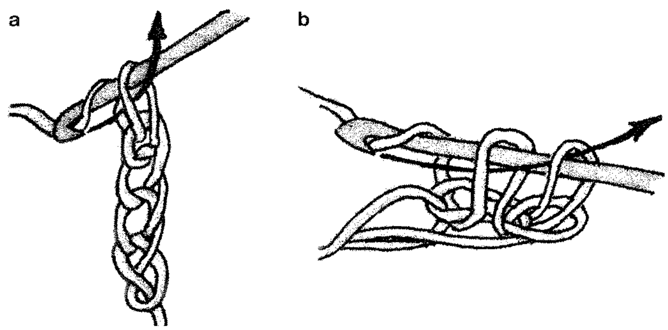

Figure 2. Crochet stitches for the hyperbolic plane.

Choose a yarn that will not stretch a lot. Every yarn will stretch a little, but you need one that will keep its shape. That's it! Now you are ready to start the stitches:

1. Make your **beginning chain stitches** (Figure 2a). (Topologists may recognize that as the stitches in the Fox-Artin wild arc!) About 20 chain stitches for the beginning will be enough.
2. For the **first stitch in each row**, insert the hook into the 2nd chain from the hook. Take yarn over and pull through chain, leaving 2 loops on hook. Take yarn over and pull through both loops. One single crochet stitch has been completed. (Figure 2b.)
3. For **the next *N* stitches**, proceed exactly like the first stitch, except insert the hook into the next chain (instead of the 2nd).
4. **For the $(N + 1)$st stitch**, proceed as before, except insert the hook into the same loop as the *N*th stitch.
5. **Repeat Steps 3 and 4** until you reach the end of the row.
6. **At the end of the row**, before going to the next row, do one extra chain stitch.
7. **When you have the model as big as you want**, you can stop, just by pulling the yarn through the last loop.

Be sure to crochet fairly tight and even. That's all you need from crochet basics. Now you can make your hyperbolic plane. You have to increase (by the above procedure) the number of stitches from one row to the next in a constant ratio, N to $N + 1$—the ratio determines the radius of the hyperbolic plane (corresponding to ρ in the former construction). You can experiment with different ratios, *but* not in the same model. You will get a hyperbolic plane *only* if you increase the number of stitches in the same ratio all the time.

Crocheting will take some time, but later you can work with this model without worrying about destroying it. The completed product is depicted in Figure 3.

A polyhedral annular hyperbolic plane

A polyhedral version of the annular hyperbolic plane can be constructed out of equilateral triangles by putting 6 triangles together at half the vertices and 7 triangles together

Figure 3. A crocheted annular hyperbolic plane.

at the others. (If we were to put 6 triangles together at every vertex, then we would get the Euclidean plane.) The precise construction can be described in three different (but, in the end, equivalent) ways:

1. Construct polyhedral annuli as in Figure 4, and then tape them together as with the annular hyperbolic plane.
2. You can construct two annuli at a time by using the shape in Figure 5 and taping one to the next by joining: $a \to A$, $b \to B$, $c \to C$.
3. The quickest way is to start with many strips, as pictured in Figure 6a. These strips can be as long as you wish. Then join four of the strips together as in Figure 6b using 5 additional triangles. Next, add another strip every place there is a vertex with 5 triangles and a gap (as at the marked vertices in Figure 6b). Every time a strip is added, an additional vertex with 7 triangles is formed.

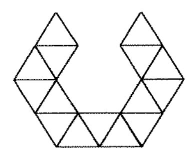

Figure 4. Polyhedral annulus.

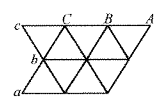

Figure 5. Shape to make two annuli.

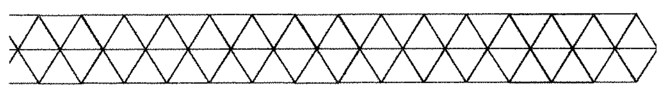

Figure 6a. Strips.

The center of each strip runs perpendicular to each annulus, and you can show that each of these curves (the center lines of the strip) is geodesic because they all have global reflection symmetry. This model has the advantage of being constructible more easily than the two models above; however, one cannot make better and better approximations by decreasing the size of the triangles. This is true because at each sevenfold vertex the cone angle is $(7 \times 60°) = 420°$, no matter what the size of the triangles, and the radius of the polyhedral annulus will decrease because it is about $1\frac{1}{2}$ times the side length of the triangles (see Figure 4), whereas the hyperbolic plane locally looks like the Euclidean plane (360°).

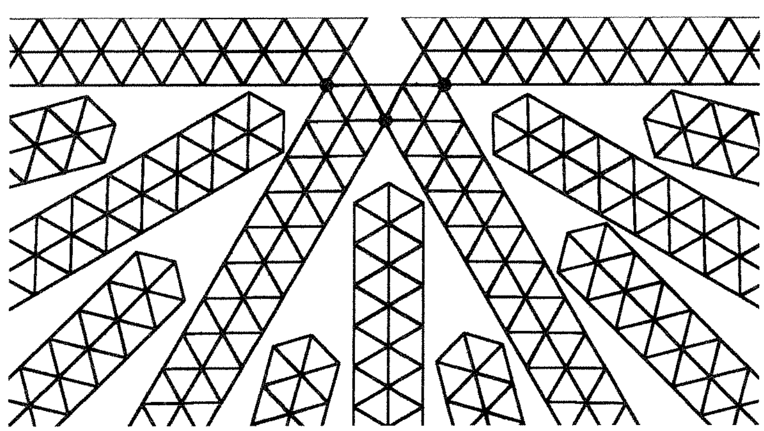

Figure 6b. Forming the polyhedral annular hyperbolic plane.

The hyperbolic soccer ball

Polyhedral models of the hyperbolic plane can also be constructed from equilateral triangles by putting 7 triangles at every vertex (the {3,7} model) or, dually, by putting 3 regular heptagons (7-gons) together at every vertex (the {7,3} model). These are difficult to use in practice because they are "pointy" with cone angles at the vertices of 420° or 385.7 . . .°. In addition, their radii are small (about the length of a side), and it is not convenient to describe the annuli and related coordinates.

Since the first version of this paper was written, Keith Henderson, David's son, showed us a better polyhedral model, which he named the *hyperbolic soccer ball.* The hyperbolic soccer ball construction is related to the {3,7} model in the sense that if a neighborhood of each vertex in the {3,7} model is replaced by a heptagon (7-sided form), then the remaining portion of each triangle is a hexagon. If you use regular heptagons and regular hexagons, then each heptagon is surrounded by seven hexagons; and two hexagons and one heptagon come together around each vertex (see Figure 7). This is the hyperbolic soccer ball. An ordinary soccer ball (outside the USA, called a "football") is constructed by using pentagons surrounded by five hexagons; and (especially if made from leather that stretches a little) is a good polyhedral approximation of the sphere. The plane can be tiled by hexagons, each surrounded by six other hexagons.

Because a heptagon has interior angles with $5\pi/7$ radians (= 128.57 . . .°), the vertices of this construction have cone angles of 368.57 . . .° and thus are much smoother than the {3,7} and {7,3} polyhedral constructions. The finished product has a nice appearance if you make the heptagons a different color from the hexagons. As with any polyhedral construction, it is not possible to get closer and closer approximations to the hyperbolic plane by changing the size of the hexagons and heptagons, and again there is no convenient way to see the annuli.

The hyperbolic soccer ball also has a radius ρ that is large enough to be used conveniently. To calculate the radius, we first tile the hyperbolic soccer ball by congruent triangles (see the triangle marked in Figure 7), which each contain a vertex of the hyperbolic soccer ball, where the curvature of the hyperbolic soccer ball is concentrated. We can then use the fact (which we prove at the end of this paper) that in the hyperbolic plane the area of a triangle is given by

$$A(\Delta) = \left(\pi - \sum \alpha_i\right)\rho^2$$

where the α_i are the interior angles of the triangle. The triangles in the tiling have angles ($\pi/3$, $\pi/3$, $2\pi/7$), and their areas can be easily calculated (using ordinary geometry) to be $(1.3851\ldots)s^2$, where s is the length of the sides of hexagons and heptagons. From this we calculate that the radius of the hyperbolic soccer ball is $\rho = (3.042\ldots)s$. For comparison, the radius of a spherical soccer ball is $(2.404\ldots)s$, which can be calculated in a similar way.

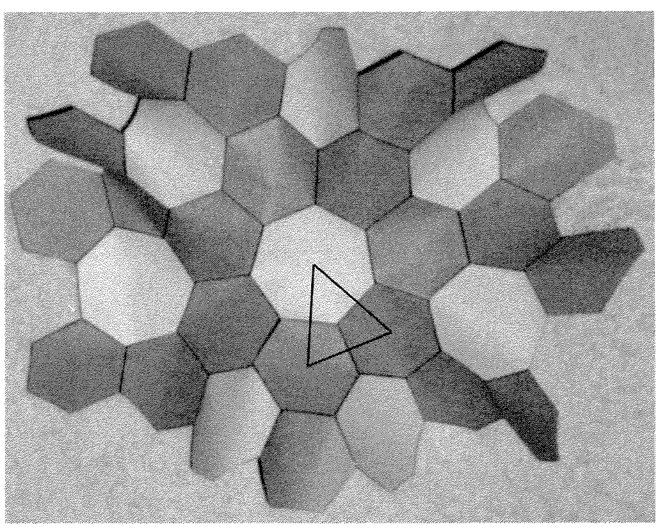

Figure 7. The hyperbolic soccer ball.

Hyperbolic planes of different radii (curvature)

Note that the construction of an annular hyperbolic plane is dependent on ρ (the radius of the annuli), which can be called the *radius of the hyperbolic plane.* As in the case of spheres, we get different hyperbolic planes depending on the value of ρ. In Figure 8 a, b, and c there are crocheted hyperbolic planes with radii approximately 4 cm, 8 cm, and 16 cm. These photos were all taken from approximately the same perspective, and in each picture there is a centimeter rule to indicate the scale.

Note that as ρ increases the hyperbolic plane becomes flatter and flatter (has less and less curvature). For both the sphere and the hyperbolic plane, as ρ goes to infinity they become indistinguishable from the ordinary flat (Euclidean) plane. We will show below that the Gaussian curvature of the hyperbolic plane is $-1/\rho^2$. So it makes sense to call this ρ the radius of the hyperbolic plane, in agreement with spheres, where a sphere of radius ρ has Gaussian curvature $1/\rho^2$.

How Do We Know that We Obtain the Hyperbolic Plane?

Why is it that the intrinsic geometry of an annular hyperbolic plane is a hyperbolic plane? The answer, of course, depends on what is meant by "hyperbolic plane." There are

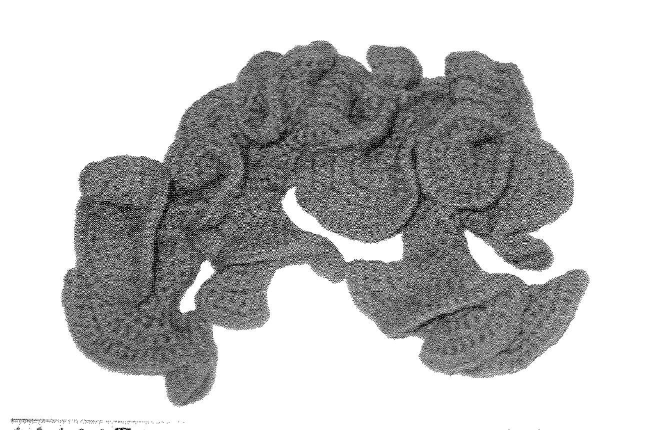

Figure 8a. Hyperbolic plane with $\rho \approx 4$ cm.

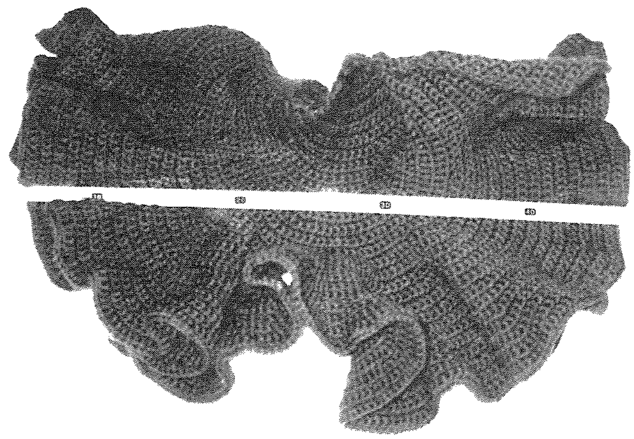

Figure 8b. Hyperbolic plane with $\rho \approx 8$ cm.

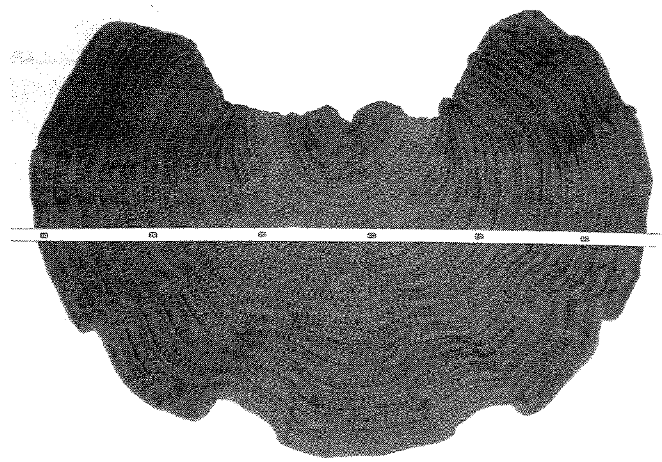

Figure 8c. Hyperbolic plane with $\rho \approx 16$ cm.

four main ways of describing the hyperbolic plane; we hope one of these is your favorite:

1. A hyperbolic plane satisfies all the postulates of Euclidean geometry except for *Euclid's Fifth (*or *Parallel) Postulate.*
2. A hyperbolic plane has the same local (intrinsic) geometry as the *pseudosphere*.
3. A hyperbolic plane is a simply connected complete *Riemannian manifold with constant negative Gaussian curvature.*
4. A hyperbolic plane is described by the *upper half-plane model.*

The italicized terms will be explained as we deal with each description in the sections that follow. But first we consider natural coordinates that we will find useful.

Intrinsic geodesic coordinates

Let ρ be the fixed inner radius of the annuli, and let H_δ be the approximation of the annular hyperbolic plane constructed, as above, from annuli of radius ρ and thickness δ. On H_δ pick the inner curve of any annulus, calling it the base curve, pick a positive direction on this curve, and pick any point on this curve and call it the origin O. We can now construct an (intrinsic) coordinate system $\mathbf{x}_\delta : \mathbf{R}^2 \to H_\delta$ by defining $\mathbf{x}_\delta(0, 0) = O$, $\mathbf{x}_\delta(w, 0)$ to be the point on the base curve at a distance w from O, and $\mathbf{x}_\delta(w, s)$ to be the point at a distance s from $\mathbf{x}_\delta(w, 0)$ along the radial (along the radii of each annulus) curve through $\mathbf{x}_\delta(w, 0)$, where the positive direction is chosen to be in the direction from outer to inner curve of each annulus (see Figure 9). The reader can easily check that this coordinate map is one-to-one and onto (if you were to crochet indefinitely). Let $\mathbf{x} = \lim_{\delta \to 0} \mathbf{x}_\delta : \mathbf{R}^2 \to H^2$, the annular hyperbolic plane.

Note that each coordinate map $\mathbf{x}_\delta$ induces a metric, d_δ, on $\mathbf{R}^2$ by defining $d_\delta(p, q)$ to be the (intrinsic) distance between $\mathbf{x}_\delta(p)$ and $\mathbf{x}_\delta(q)$ in H_δ. Those readers who desire a more formal description of the limit can check that, in the limit as $\delta \to 0$, the metrics d_δ converge to a metric d on $\mathbf{R}^2$, and this defines the annular hyperbolic plane as $\mathbf{R}^2$ with a special metric. In fact, this process also defines a Riemannian metric, but this will be easier to see after we show the connections with the upper half-plane model.

What can we experience about hyperbolic geodesics and isometries?

The following facts were observed by our students during one class period in which, working in small groups, they explored for the first time the crocheted hyperbolic plane.

The radial curves are geodesics with reflection symmetry. The radial curves (curves that run radially across each annulus) have intrinsic reflection symmetry in each H_δ because of the symmetry in each annulus and the fact that the radial curves intersect the bounding curves at right angles. These reflection symmetries carry over in the limit to the annular hyperbolic plane. Such bilateral symmetry is the basis of our intuitive notion of straightness (see Chapters 1 of references [3] and [4] for more details), and thus we can conclude that these radial curves are geodesics (intrinsically straight curves) on the annular hyperbolic plane and that reflection through these curves is an isometry.

The radial geodesics are asymptotic. Looking at our hyperbolic surfaces, we see the radial geodesics getting closer and closer in one direction and diverging in the other direction. In fact, let λ and μ be two of the radial geodesics in H_δ. The distance between these radial geodesics changes by $\rho/(\rho + \delta)$ every time they cross one annulus. (Remember, the annuli all have the same radii.) If we cross n strips, then the distance in H_δ between λ and μ at a distance $c = n\delta$ from the base curve is:

$$d\left(\frac{\rho}{\rho + \delta}\right)^n = d\left(\frac{\rho}{\rho + \delta}\right)^{c/\delta}.$$

Now take the limit as $\delta \to 0$ to show that the distance between λ and μ on the annular hyperbolic plane is:

$$d\,\exp(-c/\rho). \tag{1}$$

Asymptotic geodesics never happen on a Euclidean plane or on a sphere.

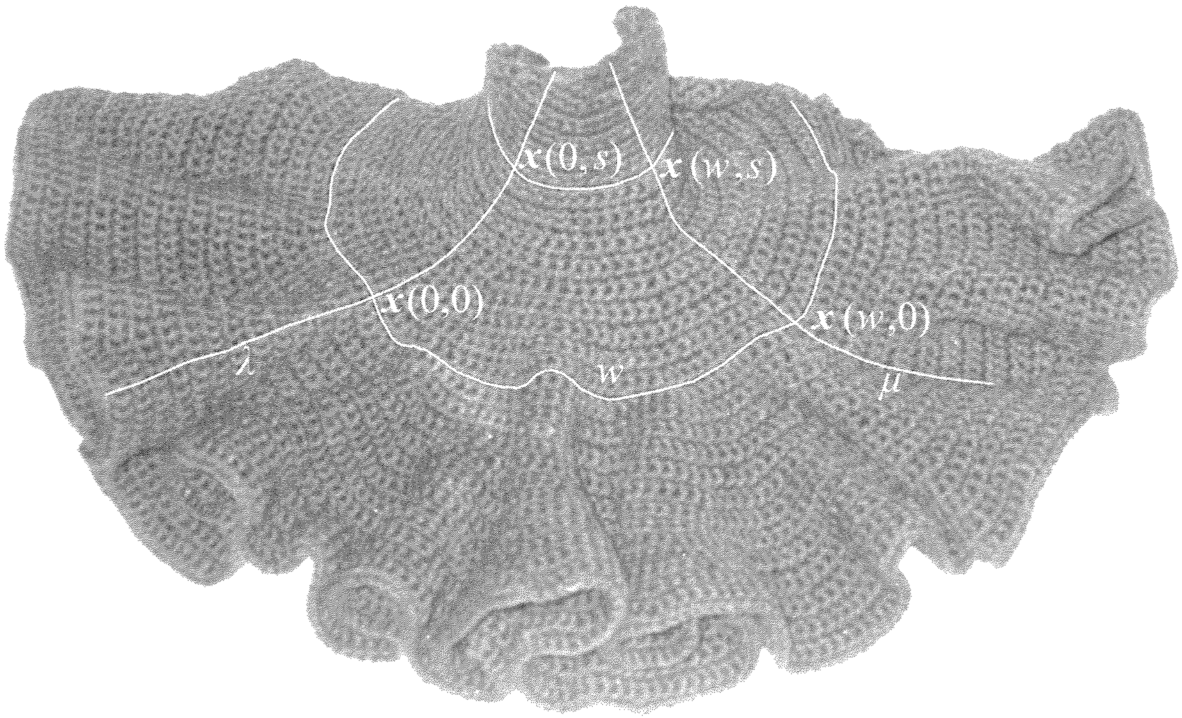

Figure 9. Geodesic coordinates on an annular hyperbolic plane.

There is an isometry that preserves the annuli. Because reflections through radial geodesics are isometries that preserve each annulus, the composition of two such reflections must also be an isometry that preserves each annulus. A brief consideration of what happens on a given annulus should convince us that this isometry shifts the annulus along itself. In the plane we would call such an isometry a rotation (about the center of the annulus). But, on the annular hyperbolic plane, an annulus has no center and the isometry has no fixed point because the radial geodesics (which are perpendicular to the annulus) do not intersect. Also, we do not want to call this isometry a *translation* because there is no geodesic that is preserved by the isometry. So, this is a type of isometry that we have not met before on the plane. Such isometries are traditionally called *horolations*, and annular curves are traditionally called *horocycles*. Horolations can be thought of as rotations about a point at infinity (since the radial geodesics are asymptotic), and the horocycles can be thought of as circles with infinite radius.

Other geodesics can be found in approximate intuitive ways.

- Hold two points of the hyperbolic surface between the index finger and thumb on your two hands. Now pull gently—a geodesic segment (with its reflection symmetry) should appear between the two points. This is using the property that a geodesic is locally the shortest path.
- Fold the surface to a crease with bilateral symmetry.
- You can lay a (straight) ribbon on the surface and it will follow a geodesic. This *Ribbon Test* for geodesics on surfaces is discussed further (with proofs) in reference [3], Problems 3.4 and 7.6.

The following properties of geodesics can be easily experienced by playing with the annular hyperbolic plane. These properties can be rigorously confirmed later by using the upper half-plane model.

G1. *Every pair of points is joined by a unique geodesic.*

G2. *Two geodesics intersect no more than once.*

G3. *Every geodesic segment has a geodesic perpendicular bisector.*

G4. *Every angle (between two geodesics) has a geodesic angle bisector.*

G5. *Each non-radial geodesic is tangent to one annulus, and then, as you travel in both directions from that point, the geodesic approaches being perpendicular to the annuli that it crosses on the way to infinity.*

Connections to Euclid's postulates

Euclid's five postulates in modern wording are:

P1. *A (unique) straight line may be drawn from any point to any other point.*

P2. *Every limited straight line can be extended indefinitely to a (unique) straight line.*

P3. *A circle may be drawn with any center and any radius.*

P4. *All right angles are equal.*

P5. *If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which the angles are less than two right angles.*

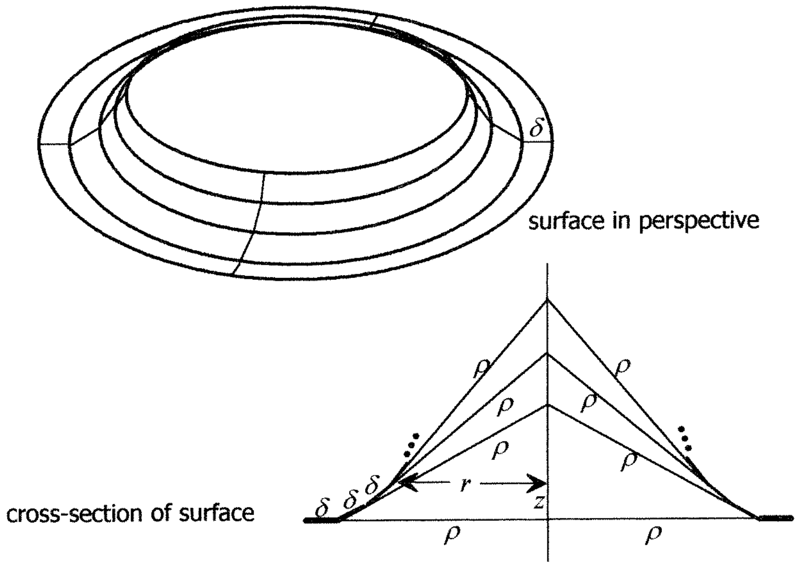

Figure 10. Hyperbolic surface of revolution.

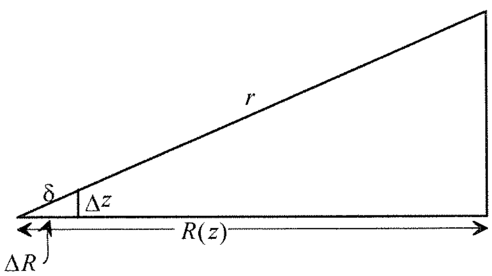

Figure 11. Relating $R(z)$, ρ, Δz, and ΔR.

We will have to wait for the analytic power of the upper half-plane model to confirm rigorously these properties for the annular hyperbolic plane, but we can give intuitive arguments now. It is easy to convince yourself that the first three postulates are true by playing with the annular hyperbolic plane, but the other two take some more thought.

All right angles are equal. What does this postulate mean? How is it possible to imagine right angles that are not equal? To see this we must look at Euclid's definition of "right angle":

> *When a straight line intersects another straight line such that the adjacent angles are equal to one another, then the equal angles are called right angles.*

By this definition, the right angles at a vertex of a polyhedron are less than 90° and thus any polyhedron can not satisfy Euclid's Fourth Postulate. To show that the annular hyperbolic plane satisfies this postulate, consider a right angle α at the point P defined by the lines l_P and m_P and another right angle β at Q defined by l_Q and m_Q. Then, by reflecting R in the perpendicular bisector (see G3) of the line segment $\overline{PQ}$ (see **G1**), the point P is taken to the point Q; one or two more reflections through the bisectors (see **G4**) of the angles defined by the sides of $R(\alpha)$ and β will eventually bring the reflected image of α into coincidence with β.

Euclid's Fifth Postulate. Consider two radial geodesics intersected by the geodesic l determined by intersections of these radial geodesics with a given annulus curve. The radial geodesics do not intersect, even though it is clear that they make angles on the same side of l that are each less than a right angle. Thus Euclid's Fifth Postulate does not hold on the annular hyperbolic plane.

In many treatments of axiomatic geometry, Euclid's Fifth Postulate is replaced by

(Playfair's) Parallel Postulate: *Given a line l and a point P not on l, there is a unique line through P that is parallel to l.*

Since any two geodesics (great circles) on a sphere intersect, it is clear that Euclid's Fifth Postulate is true on a sphere while Playfair's Postulate is not true, contrary to the statements in many books that the two are equivalent. The correct statement is that they are equivalent *in the presence of all the other postulates.*

Connection to the pseudosphere

Take the annulus whose inner edge is the base curve and embed it isometrically in the x–y plane as a complete annulus with center at the origin. Now attach to this annulus portions of the other annuli as indicated in Figure 10. Note that the second and subsequent annuli form truncated cones.

Let the vertical axis be the z-axis; then at each z we have the picture in Figure 11.

Thus

$$\frac{\Delta R}{\Delta z} = \frac{-R(z)}{\sqrt{(\rho+\delta)^2 - R(z)^2}}.$$

In the limit as δ (and ΔR and Δz) go to zero, we get

$$\frac{dR}{dz} = \frac{-R(z)}{\sqrt{\rho^2 - R(z)^2}}. \tag{2}$$

We can get the same differential equation by using (1) above, which implies that the circle at height z has circumference $2\pi\rho e^{-s/\rho}$, where s is the arc length along the surface from $(0, r)$ to $(z, R(z))$. We can solve this differential equation explicitly for z:

$$z = \sqrt{\rho^2 - R^2} - \rho \ln \left| \frac{\rho + \sqrt{\rho^2 - R^2}}{R} \right|.$$

Here z is a continuously differentiable function of R and the derivative (for $z \neq 0$) is never zero, hence R is also a

Figure 12. Crocheted pseudosphere.

continuously differentiable function of z. Because R is never zero, we can conclude that this hyperbolic surface of revolution is a smooth surface (traditionally called the *pseudosphere*). Thus,

THEOREM: *The pseudosphere is locally isometric to the annular hyperbolic plane.*

We can also crochet a pseudosphere by starting with 5 or 6 chain stitches and continuing in a spiral fashion, increasing as when crocheting the hyperbolic plane (see Figure 12). Note that, when you crochet beyond the annular strip that lies flat and forms a complete annulus, the surface begins to form ruffles and is no longer a surface of revolution. In fact, it appears that it is not even differentiable where the ruffles start, for the "top ridge" of the ruffles (see Figure 12) appears to be straight and thus not tangent to the plane of the complete annulus.

Connections to Riemannian manifolds with constant negative Gaussian curvature

If a surface is differentiably embedded into 3-space by an isometry whose first and second derivatives are continuous (C^2), then the surface is said to be a Riemannian manifold. At a given point P on the surface, call *the normal direction* one of the two directions that are perpendicular to the surface at that point. The *normal curvature* at a point of a curve on the surface is defined to be the component of the curvature of the curve that is in the normal direction. The collection of all normal curvatures of all the (smooth) curves through P has a maximum and a minimum value. These extremal values of the normal curvature are the principal curvatures (and can be shown to be the normal curvatures of two curves that are perpendicular at P). The *Gaussian curvature* of the surface at P is defined to be the product of these two principal curvatures.

The pseudosphere is a Riemannian surface, and at each point $[z, R(z), \theta]$ the principal curvatures are the normal curvatures of generating curves, $z \mapsto R(z)$ and the circle $\theta \mapsto [R(z), \theta]$. The curvature of the first curve is

$$\frac{-R''(z)}{[1 + (R'(z))^2]^{3/2}}$$

and is perpendicular to the surface, and thus is also ($\pm$) normal curvature. The curvature of the circle is $1/R(z)$, which must be projected onto the direction perpendicular to the surface, giving the normal curvature as

$$\frac{1}{R(z)\sqrt{1 + (R'(z))^2}}.$$

We do not have a formula for R, but we do have a formula (2) for $R'(z)$. The Gaussian curvature is then the product of these two normal curvatures, which you can check [using (2)] is $-1/\rho^2$; the minus sign occurs because the two normal curvatures are in opposite directions. Thus, the pseudosphere has constant negative Gaussian curvature.

Gauss's famous *Theorema Egregium* states that the Gaussian curvature is independent of the (C^2) embedding, hence is an intrinsic property of the surface. Thus, since the annular hyperbolic plane is locally isometric to the pseudosphere, we can say it also has constant negative Gaussian curvature. Most differential geometry texts give intrinsic methods for determining the Gaussian curvature, which can be applied directly to the annular hyperbolic plane (see [3], Problem 7.7, for two such methods). Note that in the crocheted pseudosphere (Figure 12) there are points that apparently have no tangent planes and thus no normal direction, and therefore it is not possible to define (at these points) the principal curvatures. In addition, the result of N. V. Efimov [2] already discussed shows that, no matter how the annular hyperbolic plane is placed in 3-space, if it is extended enough it cannot be C^2 embedded (and thus cannot have principal curvatures at all points).

Connection to the upper half-plane model

As shown above, the coordinate map $\mathbf{x}$ preserves (does not distort) distances along the (vertical) 2nd coordinate curves, but at $\mathbf{x}(a, b)$ the distances along the 1st coordinate curve are distorted by the factor of $\exp(-b/\rho)$ when compared to the distances in $\mathbf{R}^2$. To be more precise:

DEFINITION: *Let* $\mathbf{y}: A \to B$ *be a map from one metric space to another, and let* $t \mapsto \lambda(t)$ *be a curve in A. Then, the* distortion *of* $\mathbf{y}$ *along* λ *at the point* $p = \lambda(0)$ *is defined as:*

$$\lim_{x \to 0} \frac{\text{arc length along } \mathbf{y}(\lambda) \text{ from } \mathbf{y}[\lambda(x)] \text{ to } \mathbf{y}[\lambda(0)]}{\text{arc length along } \lambda \text{ from } \lambda(x) \text{ to } \lambda(0)}.$$

We seek a change of coordinates that will distort distances equally in both directions. The reason for seeking this change is that if distances are distorted to the same degree in both coordinate directions, then the map will preserve angles. (We call such a map *conformal.*)

We cannot hope to have zero distortion in both coordinate directions (if there were no distortion then the chart would be an isometry), so we try to make the distortion in the 2nd coordinate direction the same as the distortion in the 1st coordinate direction. After a little experimentation, we find that the desired change is

$$\mathbf{z}(x, y) = \mathbf{x}[x, \rho \ln(y/\rho)]$$

with the domain of $\mathbf{z}$ being the upper half-plane

$$\mathbf{R}^{2+} \equiv \{(x, y) \in \mathbf{R}^2 \,|\, y > 0\}$$

where $\mathbf{x}$ is the geodesic coordinate map defined above. This is the usual *upper half-plane model* of the hyperbolic plane, thought of as a map of the hyperbolic plane in the same way that we use planar maps of the spherical surface of the earth.

LEMMA: *The distortion of* $\mathbf{z}$ *along both coordinate curves*

$$x \to \mathbf{z}(x, b) \text{ and } y \to \mathbf{z}(a, y)$$

at the point $\mathbf{z}(a, b)$ *is* ρ/b.

PROOF. We now focus on the point $\mathbf{z}(a, b) = \mathbf{x}(a, \rho\ln(b/\rho))$. Along the first coordinate curve, $x \to \mathbf{z}(x, b) = \mathbf{x}(x, \rho \ln(b/\rho))$, the arc length from $\mathbf{x}(a, c)$ to $\mathbf{x}(x, c)$ is $|x - a| \exp(-c/\rho)$ by (1) above. Thus, we can calculate the distortion:

$$\lim_{x \to a} \frac{|x - a| \exp[-[\rho \ln(b/\rho)]/\rho]}{|x - a|} = \rho/b.$$

Now, look at the second coordinate curve, $y \to \mathbf{z}(a, y) = \mathbf{x}(a, \rho \ln(y/\rho))$. Along this coordinate curve (a radial geodesic) the speed is not constant; but, since the second coordinate of $\mathbf{x}$ measures arc length, the arc length from $\mathbf{z}(a, y) = \mathbf{x}(a, \rho \ln(y/\rho))$ to $\mathbf{z}(a, b) = \mathbf{x}(a, \rho \ln(b/\rho))$ is

$$|\rho \ln(y/\rho) - \rho \ln(b/\rho)|$$

and the distortion is

$$\lim_{y \to b} \frac{|\rho \ln(y/\rho) - \rho \ln(b/\rho)|}{|y - b|} = \rho \left| \lim_{y \to b} \frac{\ln(y/\rho) - \ln(b/\rho)}{y - b} \right|$$

$$= \rho \left| \frac{d}{dy} \ln(y/\rho) \right|_{y=b} = \rho/b.$$

In the above situation, we call these distortions the *distortion of the map* $\mathbf{z}$ *at the point* p and denote it $\mathbf{dist}(\mathbf{z})(p)$. Thus,

$$\mathbf{dist}(\mathbf{z})((a, b)) = \rho/b$$

Hyperbolic Isometries and Geodesics

We have seen that there are reflections in the annular hyperbolic plane about the radial geodesics, but we saw the existence of other reflections and geodesics only approximately. However, we were able to see that non-radial geodesics appear to be tangent to one annulus and then in both directions from that point to approach being perpendicular to the annuli. To assist us in looking at transformations of the annular hyperbolic space, we use the upper half-plane model. As the annuli correspond to horizontal lines in the upper half-plane model, geodesics should then be curves that start and end perpendicular to the boundary x-axis. Semicircles with centers on the x-axis are such curves, and we can show directly that they are geodesics with bilateral symmetry. In particular, we can show directly that inversion in a semicircle corresponds to a reflection isometry in the annular hyperbolic plane.

DEFINITION: An *inversion with respect to a circle* Γ is a transformation from the extended plane (the plane with ∞, the point at infinity, added) to itself that takes C, the center of the circle, to ∞ and vice versa and that takes a point at a distance s from the center to the point on the same ray (from the center) that is at a distance of r^2/s from the center, where r is the radius of the circle (see Figure 13). We call (P, P') an *inversive pair* because (as the reader can check) they are taken to each other by the inversion. The circle Γ is called the *circle of inversion.*

Inversions have the following well-known properties (see reference [1], Chapter 5, and reference [4], Chapter 14):

- Inversions are conformal.
- Inversions take circles not passing through the center of inversion to circles.
- Inversions take circles passing through the center of inversion to straight lines not through the center of inversion.

If $\mathbf{f}$ is a transformation taking the upper half-plane $\mathbf{R}^{2+}$ to itself, then consider the diagram

$$\begin{array}{ccc} H^2 & \xrightarrow{\mathbf{g}} & H^2 \\ \mathbf{z}^{-1} \downarrow & & \uparrow \mathbf{z} \\ \mathbf{R}^{2+} & \xrightarrow[\mathbf{f}]{} & \mathbf{R}^{2+} \end{array}$$

We call $\mathbf{g} = \mathbf{z} \circ \mathbf{f} \circ \mathbf{z}^{-1}$ the transformation of H^2 that corresponds to $\mathbf{f}$. We will call $\mathbf{f}$ an *isometry of the upper half-plane model* if the corresponding $\mathbf{g}$ is an isometry of the annular hyperbolic plane.

THEOREM: *Let* $\mathbf{f}$ *be the inversion in a circle whose center is on the* x*-axis. Then the corresponding* $\mathbf{g} = \mathbf{z} \circ \mathbf{f} \circ \mathbf{z}^{-1}$ *has distortion* 1 *at every point and is thus an isometry.*

PROOF. (Refer to Figure 14.)

1. Note that each of the maps $\mathbf{z}$, $\mathbf{z}^{-1}$, $\mathbf{f}$ is conformal and has at each point a (non-zero) distortion that is the same

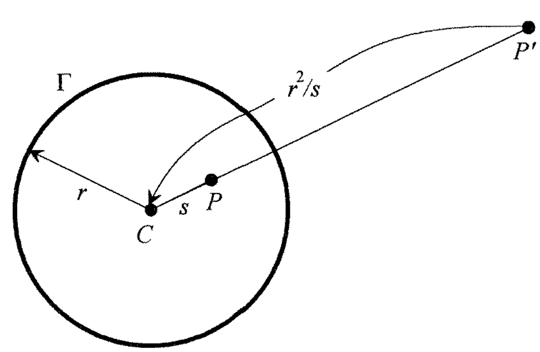

Figure 13. Inversion with respect to a circle.

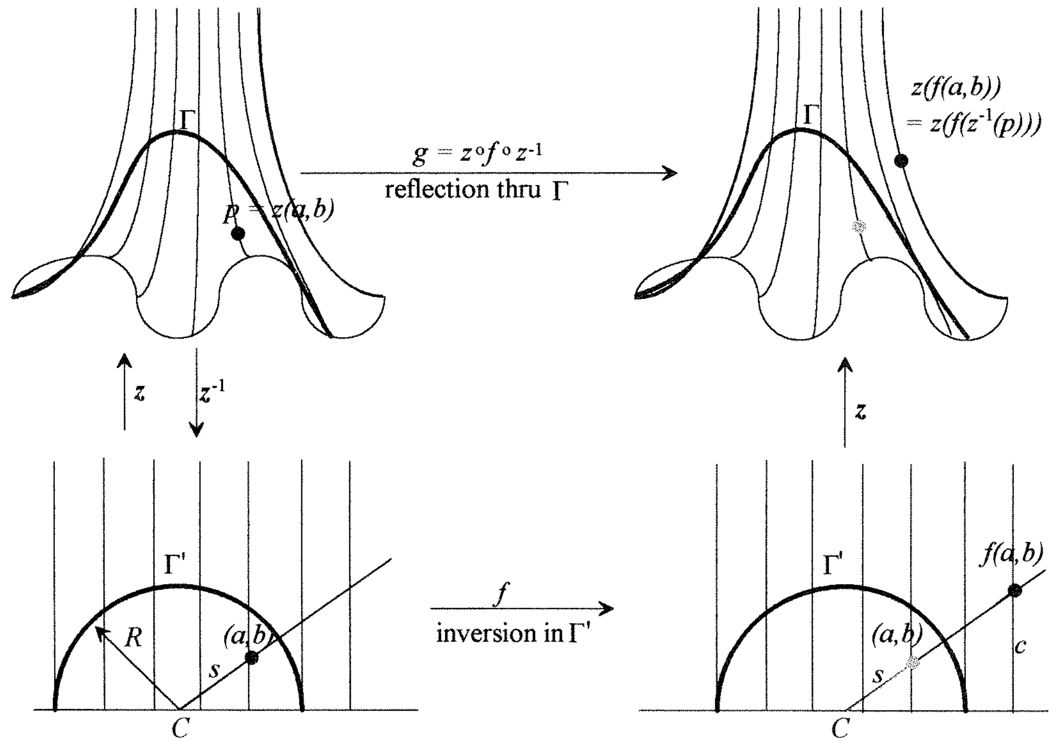

Figure 14. Hyperbolic reflections correspond to inversions.

for all curves at that point. Using the definition of distortion, the reader can easily check that

$$\mathbf{dist}(\mathbf{g})(p) = \mathbf{dist}(\mathbf{z})((\mathbf{f} \circ \mathbf{z}^{-1})(p)) \times \mathbf{dist}(\mathbf{f})(\mathbf{z}^{-1}(p)) \times \mathbf{dist}(\mathbf{z}^{-1})(p).$$

2. If $\mathbf{z}(a, b) = p$, then, using (1),

$$\mathbf{dist}(\mathbf{z}^{-1})(p) = 1/[\mathbf{dist}(\mathbf{z})((\mathbf{z}^{-1}(p))] = b/\rho.$$

3. Let r be the radius of the circle C which defines $\mathbf{f}$, and let s be the distance from the center of C to $(a, b) = \mathbf{z}^{-1}(p)$. The inversion being conformal, the distortion is the same in all directions. Thus, we need only check the distortion along the ray from C, the center of circle, through p. The reader can check that this distortion

$$\mathbf{dist}(\mathbf{f})((a, b)) = r^2/s^2.$$

One way to do this is to note that, in this case, the distortion is the speed (at s) of the curve $t \mapsto r^2/t$.

4. By (1), $\mathbf{dist}(\mathbf{z})(\mathbf{f}(\mathbf{z}^{-1}(p)) = \rho/c$, where c is the y-coordinate of $\mathbf{f}(\mathbf{z}^{-1}(p)) = \mathbf{f}(a, b)$. To determine c, look at Figure 14. By similar triangles, $s/b = (r^2/s)/c$. Thus $c = b(r^2/s^2)$ and

$$\mathbf{dist}(\mathbf{z})((\mathbf{f} \circ \mathbf{z}^{-1})(p)) = \frac{\rho s^2}{br^2}.$$

5. Putting everything together, we now have

$$\mathbf{dist}(\mathbf{g})(p) = \frac{\rho s^2}{br^2}\,\frac{r^2}{s^2}\,\frac{b}{\rho} = 1.$$

Since this is true at all points p, the map $\mathbf{g}$ must be an isometry of the annular hyperbolic plane.

We call these inversions through semicircles with center on the x-axis (or the corresponding transformations in the annular hyperbolic plane) *hyperbolic reflections*. Thus the images of the semicircles in the upper half-plane have bilateral symmetry and so are intrinsically straight (geodesics).

We have established that the annular hyperbolic plane is the same as the usual upper half-plane model of the hyperbolic plane. The usual analysis of the hyperbolic plane can

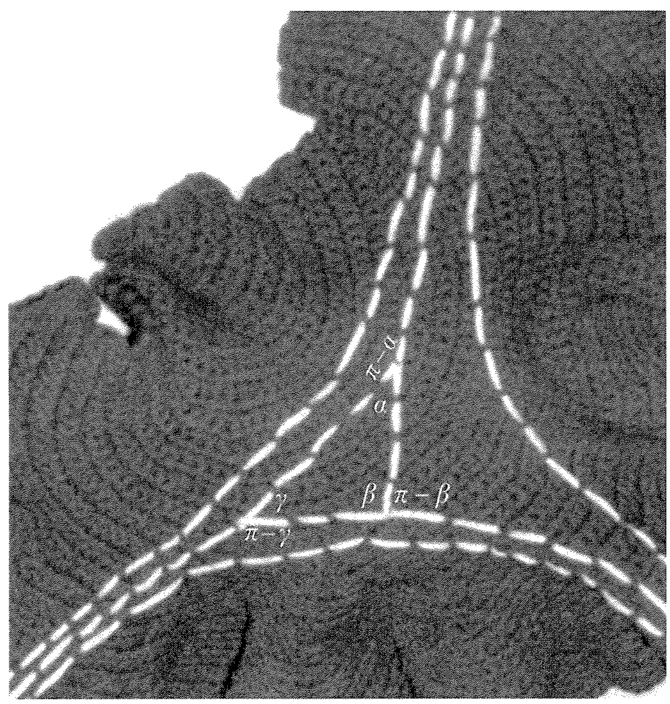

Figure 15. Triangle with an ideal triangle and three 2/3-ideal triangles.

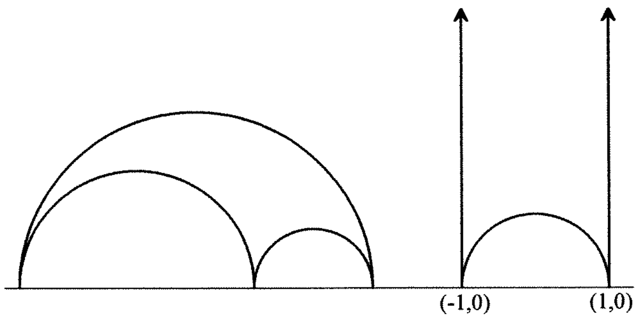

Figure 16. Ideal triangles in the upper half-plane model.

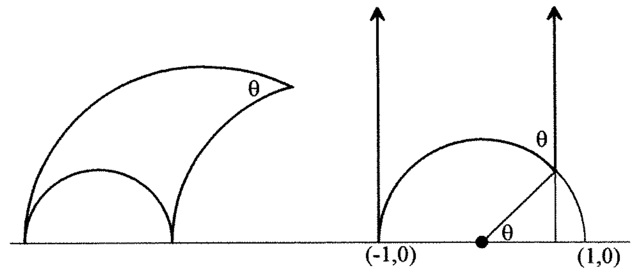

Figure 17. 2/3-ideal triangles in the upper half-plane model.

now be considered as analysis of the intrinsic geometry of the annular hyperbolic plane. We give only one example here because it results in the interesting formula πr^2.

Area of Hyperbolic Triangles

Given a geodesic triangle with interior angles β_i and exterior angles α_i, we extend the sides of the triangle as indicated in Figure 15.

The three extra lines are geodesics that are asymptotic at both ends to an extended side of the triangle. It is traditional to call the region enclosed by these three extra geodesics an *ideal triangle*. In the annular hyperbolic plane these are not actually triangles because their vertices are at infinity. In Figure 15 we see that the ideal triangle is divided into the original triangle and three "triangles" that have two of their vertices at infinity. We call a "triangle" with two vertices at infinity a *2/3-ideal triangle*. You can use this decomposition to determine the area of the hyperbolic triangle. First we must determine the areas of ideal and 2/3-ideal triangles.

It is impossible to picture the whole of an ideal triangle in an annular hyperbolic plane, but it is easy to picture ideal triangles in the upper half-plane model. In the upper half-plane model an *ideal triangle* is a triangle with all three vertices either on the x-axis or at infinity (see Figure 16).

At first glance it appears that there must be many different ideal triangles; however:

THEOREM: *All ideal triangles on the same hyperbolic plane are congruent.*

PROOF OUTLINE: Perform an inversion (hyperbolic reflection) that takes one of the vertices (on the x-axis) to infinity and thus takes the two sides from that vertex to vertical lines. Then apply a similarity to the upper half-plane, taking this to the standard ideal triangle with vertices (-1.0), 0, 1), and ∞ (see Figure 16).

THEOREM: *The area of an ideal triangle is* $\pi\rho^2$. (Remember, this ρ is the radius of the annuli, and equal to $\sqrt{-1/K}$, where K is the Gaussian curvature.)

PROOF: By (3), the distortion $\mathbf{dist(z)}(a, b)$ is ρ/b, and thus the desired area is

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\left(\frac{\rho}{y}\right)^2 dy\, dx = \pi\rho^2.$$

We now picture in Figure 17 *2/3-ideal triangles* in the upper half-plane model.

THEOREM: *All 2/3-ideal triangles with angle θ are congruent and have area* $(\pi - \theta)\rho^2$.

Show, using inversions, that all 2/3-ideal triangles with angle θ are congruent to the standard one at the right of Figure 17 and thus that the area is the double integral:

$$\int_{-1}^{\cos\theta}\int_{\sqrt{1-x^2}}^{\infty}\left(\frac{\rho}{y}\right)^2 dy\, dx.$$

Combining these three theorems and Figure 15 we get:

THEOREM: *The area of a hyperbolic triangle is*

$$\left(\pi - \sum\alpha_i\right)\rho^2.$$

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Daina Taimina was born and received all her formal education in Riga, Latvia. She has been teaching at the University of Latvia since 1977. Her Ph.D. thesis was in theoretical computer science, but later she moved toward geometry, history of mathematics, and mathematics education. These interests led to her being invited to the ICMI Study Conference on the Teaching of Geometry in Sicily in 1995, where she met David. She has written a book on the history of mathematics (in Latvian) and contributed to David's two recent geometry books. Currently, Daina is a visiting associate professor at Cornell University—something she could not dream about 21 years ago when between studies of automata theory she was first reading *The Mathematical Intelligencer* in Moscow's Library of Sciences & Technology (probably the only place where it was available in the former U.S.S.R.).